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STEADY-STATE HARMONIC ANTIPLANE VIBRATIONS OF A TWO-LAYER ELASTIC HALF-SPACE WITH A CYLINDRICAL CAVITY

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1. Formulation of a Boundary-Value Problem on Antiplane Steady Harmonic Vibrations. Let an elastic medium in a rectangular Cartesian coordinate system (x, y, z) occupy the region $x \geq -b$, $r = \sqrt{(x-h)^2 + y^2} \geq a$. A layer of thickness $b(-b \leq x \leq 0)$ with the parameters ρ, μ (ρ is density and μ is the shear modulus) is rigidly connected with the half-space $x \geq 0$. The half-space is characterized by the parameters ρ_1 and μ_1 and as a whole contains a horizontal cylindrical cavity of radius a with its center at the point (h, 0).

Distributed shearing forces are assigned on the boundary of the region, these forces undergoing steady harmonic oscillations over time with the frequency ω:

$$x = -b:\ \tau_{xz} = Z(y)\,e^{-i\omega t},\ r = a:\ \tau_{rz}^{(1)} = T(\varphi)\,e^{-i\omega t}. \tag{1.1}$$

Forces of rigid adhesion are assigned on the interface between the layer and half-space (x = 0), these forces determining the equality of the displacements (w(x, y)) and the shearing stresses τ_{xz}:

$$\begin{gathered} w(x, y)|_{x\to -0} = w^{(1)}(x, y)|_{x\to +0}, \\ \tau_{xz}(x, y)|_{x\to -0} = \tau_{xz}(x, y)|_{x\to +0}. \end{gathered} \tag{1.2}$$

Here and below, the superscript (1) denotes characteristics of the half-space. The motion of the medium is described by the dynamical equations of the theory of elasticity in displacements — the Lame equations [1]. We will seek to solve the formulated boundary-value problem in the class of integrable functions.

We designate the contact stresses on the interface as follows

$$x = 0:\ \tau_{xz}(0, y, t) = R(y)\,e^{-i\omega t} = \tau_{xz}^{(1)}(0, y, t). \tag{1.3}$$

In this case, we will use the method of Fourier transformation to solve the boundary-value problem for an elastic layer $-b \leq x \leq 0$ with boundary conditions (1.1), (1.3). Here, the

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expression for the amplitude of the displacement of points of the layer is

$$w(\bar{x},\bar{y})=\frac{a}{2\pi\mu}\int\limits_{\Gamma}\left(\tilde{R}(\alpha)\operatorname{ch}\sigma(\bar{x}-\bar{b})-\tilde{Z}(\alpha)\operatorname{ch}\sigma\bar{x}\right)/(\sigma\operatorname{sh}\sigma\bar{b})\,e^{-i\alpha\bar{y}}\,d\alpha, \tag{1.4}$$

where $\tilde{R}(\alpha)=F[R(y)]$; $\tilde{Z}(\alpha)=F[Z(y)]$; $\sigma=\sqrt{\alpha^2-\theta^2}$; $\theta^2=\rho\omega^2a^2/\mu$; $\bar{y}=y/a$, $\bar{x}=x/a$, $\bar{b}=b/a$ are dimensionless parameters.

The contour Γ is chosen in acordance with the principle of limiting absorption and has the following form [2]: it bends the positive singularities of the integrand function downward and the negative singularities upward. It coincides with the real axis on the remaining part.

The amplitude function of the field of displacements excited by the load (1.3) in an elastic half-space with a cavity is constructed by the superposition method:

$$w^{(1)}=w_1^{(1)}(\bar{x},\bar{y})+w_2^{(1)}(\bar{x},\bar{y}), \tag{1.5}$$

where

$$w_1^{(1)}(\bar{x},\bar{y})=-\frac{a}{2\pi\mu}\int\limits_{\Gamma}\int\limits_{-\infty}^{\infty}\frac{R_1(\eta)}{\sigma_1}\exp\left(-\sigma_1\bar{x}+i\alpha(\eta-\bar{y})\right)d\eta d\alpha; \tag{1.6}$$

$$w_2^{(1)}(\bar{x},\bar{y})=\bar{w}_2^{(1)}(\bar{r},\varphi)=\frac{a}{2\pi\mu_1\theta_1}\sum_{m=-\infty}^{\infty}\frac{H_m^{(1)}(\theta_1 r)\,e^{im\varphi}}{H_{m-1}^{(1)}(\theta_1)-\frac{m}{\theta_1}H_m^{(1)}(\theta_1)}\times$$

$$\times\int\limits_{-\pi}^{\pi}T_1(\eta)\,e^{-i\eta m}d\eta;$$

$$\sigma_1=\sqrt{\alpha^2-\theta_1^2};\ \theta_1^2=\frac{\rho_1\omega^2a^2}{\mu_1};\ \bar{r}=\sqrt{\left(\bar{x}-\frac{1}{\varepsilon}\right)^2+y^2};$$

$$\varphi=\operatorname{arctg}\frac{\bar{y}}{\bar{x}-\varepsilon^{-1}};\ \varepsilon=\frac{a}{h}.$$

Here and below, we use only dimensionless coordinates $\bar{x}$, $\bar{y}$, $\bar{r}=r/a$. Thus, the bar will be omitted. The stress functions $R_1(y)$, $T_1(\phi)$ are determined from the system of integral equations [3]:

$$R_1(y)+\frac{1}{2\pi\theta_1}\int\limits_{-\pi}^{\pi}T_1(\eta)K_1(y,\eta)\,d\eta=R(y), \tag{1.7}$$

$$T_1(\varphi)+\frac{1}{2\pi}\int\limits_{\Gamma}\int\limits_{-\infty}^{\infty}R_1(\eta)K_2(\alpha,\varphi)\,e^{i\alpha\eta}\,d\eta d\alpha=T(\varphi).$$

Here

$$K_1(y,\eta)=\sum_{m=-\infty}^{\infty}\frac{e^{im(\operatorname{arctg}(-y\varepsilon)-\eta)}}{\Delta_m(\theta_1)}\times$$

$$\times\left\{-\frac{imy}{r_0^2}H_m^{(1)}(\theta_1 r_0)-\frac{\theta_1}{\varepsilon r_0}\left(-\frac{m}{\theta_1 r_0}H_m^{(1)}(\theta_1 r_0)+H_{m-1}^{(1)}(\theta_1 r_0)\right)\right\};\ r_0=\sqrt{y^2+\varepsilon^{-2}};$$

$$K_2(\alpha,\varphi)=\exp\left(-\sigma_1(\varepsilon^{-1}+\cos\varphi)-i\alpha\sin\varphi\right)\left(\cos\varphi+\frac{i\alpha}{\sigma_1}\sin\varphi\right);$$

$$\Delta_m(\theta_1)=H_{m-1}^{(1)}(\theta_1)-\frac{m}{\theta_1}H_m^{(1)}(\theta_1).$$

The function R(y) in system (1.7) determines the distribution of contact stresses along the interface of the elastic parameters of the medium (x = 0). This function is unknown. To close the system, we use adhesion condition (1.2) and consider that the equality of the stresses (the second condition of (1.2)) is satisfied automatically by a boundary condition from auxiliary boundary-value problems (1.3). From the first condition of (1.2) we have

$$\frac{1}{\mu}\int_{\Gamma}\int_{-\infty}^{\infty}\frac{(R(\eta)\operatorname{ch}\sigma b - Z(\eta))\,e^{i\alpha(\eta-y)}}{\sigma\operatorname{sh}\sigma b}\,d\eta d\alpha + \frac{1}{\mu_1}\int_{\Gamma}\int_{-\infty}^{\infty}\frac{R_1(\eta)\,e^{i\alpha(\eta-y)}}{\sigma_1}\,d\eta d\alpha + \tag{1.8}$$

$$+\frac{1}{\mu_1\theta_1}\int_{-\pi}^{\pi}K_3(y,\eta)\,T_1(\eta)\,d\eta = 0,$$

$$K_3(y,\eta) = \sum_{m=-\infty}^{\infty}\frac{\exp\left(im\left(-\eta + \operatorname{arctg}(-y\varepsilon)\right)\right)H_m^{(1)}(\theta_1 r_0)}{\Delta_m(\theta_1)}.$$

Thus, to determine the unknown stress functions $R_1(y)$, $T_1(\phi)$, $R(y)$, we obtained a system of three integral equations (1.7), (1.8). We can describe the wave field in the medium by solving these equations with Eqs. (1.4)-(1.6).

2. Solution of the System of Integral Equations. It follows from analysis of the elements of system (1.7), (1.8) that at $\varepsilon = a/h < 1$ the operator of the system is fully continuous in the space of integrable functions. At $\varepsilon \ll 1$, similar to [3, 4], it will be compressive. This makes it possible, in constructing the solution of the system, to make effective use of the method of successive approximations, with asymptotic calculation of the integrals [5].

Let us examine the case of a cavity of relatively small radius in greater detail. We can subdivide the initial boundary-value problem (1.1), (1.2) into two parts without loss of generality.

Problem 1. The boundary of a two-layer half-space ($x = -b$) is free of stresses, while distributed forces $Z(y) \equiv 0$, $T(\phi) \neq 0$ are assigned on the cavity ($r = 1$).

Problem 2. The boundary of a two-layer half-space ($x = -b$) is loaded, while the boundary of the cavity ($r = 1$) is free of forces $Z(y) \neq 0$, $t(\phi) \equiv 0$.

Let us examine problem 1 for $\varepsilon \ll 1$. Analysis of the elements of system (1.7), (1.8) in a first approximation leads to the solution

$$R_1(y) = O\left(\sqrt{\frac{\varepsilon}{\theta_1}}\right),\quad T_1(\varphi) = T(\varphi) + O\left(\sqrt{\frac{\varepsilon}{\theta_1}}\right).$$

To calculate the second approximation, we specify the boundary conditions. We put $T(\phi) = p = \text{const}$. Considering that the Fourier transforms of the functions $Z(y)$, $R(y)$, $R_1(y)$ rather than the functions themselves enter into the representation describing the wave field in the medium, we write the following in the second approximation

$$\tilde{R}_1(\alpha) = F[R(y)] = \frac{2p\exp\left[i\left(\varepsilon^{-1}\sqrt{\theta_1^2-\alpha^2}-\pi/2\right)\right]D^-}{\theta_1 H_{-1}^{(1)}(\theta_1)D^+} + O\left(\sqrt{\frac{\varepsilon}{\theta_1}}\right), \tag{2.1}$$

$$T_1(\varphi) = p\left(1 + \sqrt{\frac{\varepsilon}{\pi\theta_1}}\,\frac{\exp\left[i\left(2\theta_1\varepsilon^{-1}+\pi/4-\theta_1\cos\varphi\right)\right]D_0^+}{H_{-1}^{(1)}(\theta_1)D_0^-}\cos\varphi\right) + O\left(\frac{\varepsilon}{\theta_1}\right),$$

$$D^{\pm} = \pm\sigma_1\mu_1\operatorname{ch}\sigma b + \mu\sigma\operatorname{sh}\sigma b,\quad D_0^{\pm} = \cos\theta b \pm i\sqrt{\frac{\rho\mu}{\rho_1\mu_1}}\sin\theta b.$$

Proceeding in a similar manner for problem 2, in the first approximation we obtain

$$R_1(y) = \frac{1}{2\pi}\int_{\Gamma}\frac{\mu_1\sigma_1}{\mu\sigma\operatorname{sh}\sigma + \mu_1\sigma_1\operatorname{ch}\sigma_1}\int_{-\infty}^{\infty}Z(\eta)\,e^{i\alpha\eta}d\eta\, e^{-i\alpha y}d\alpha,\quad T_1(\varphi) = O\left(\sqrt{\frac{\varepsilon}{\theta_1}}\right).$$

In calculating the second approximation, we specify the boundary conditions on the plane boundary, having put

$$Z(y) = \begin{cases} p, & y \in [c, d], \quad p = \text{const}, \\ 0, & y \,\overline{\in}\, [c, d]. \end{cases}$$

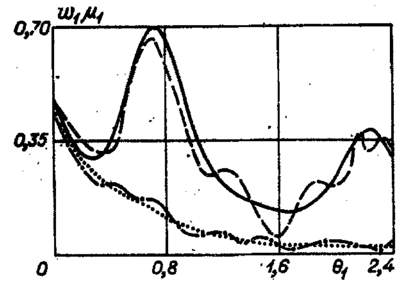

Fig. 1

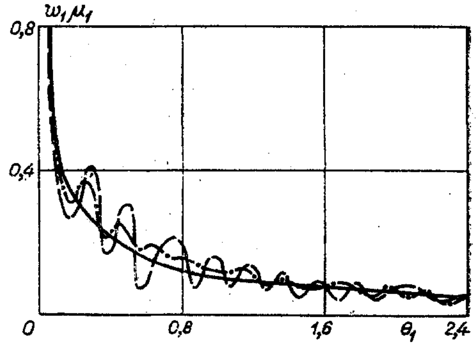

Fig. 2

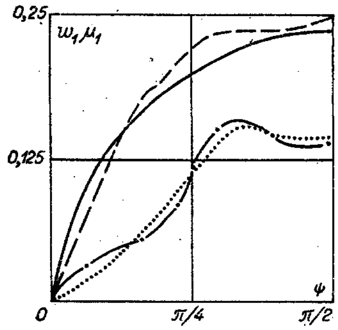

Fig. 3

In this case, calculation of the integrals in the second approximation yields

$$\widetilde{R}_1(\alpha) = \frac{\mu_1\sigma_1\widetilde{Z}(\alpha)}{D^+} + 4\sqrt{\frac{\varepsilon}{2\pi\theta_1}}\, e^{i\varepsilon^{-1}\left(\sqrt{\theta_1^2-\alpha^2}+\theta_1\right)}(d-c)\, p e^{-i\pi/4} \times$$
$$\times\left\{J_0'(\theta_1)/(2\Delta_0(\theta_1)) + \sum_{m=1}^{\infty} J_m'(\theta_1)\, T_m\left(\frac{\sqrt{\theta_1^2-\alpha^2}}{\theta_1}\right)\Big/\Delta_m(\theta_1)\right\}\times$$
$$\times D^-/(D_0^- D^+) + O\left(\frac{\varepsilon}{\theta_1}\right), \tag{2.2}$$
$$T_1(\varphi) = -\sqrt{\frac{\varepsilon\theta_1}{2\pi}}\exp\left(i\left(\theta_1\cos\varphi + \theta_1\varepsilon^{-1} - \pi/4\right)\right)/\Delta_0(\theta_1)\times$$
$$\times\cos\varphi\cdot p(d-c) + O\left(\frac{\varepsilon}{\theta_1}\right).$$

If necessary, the construction of successive approximations can be continued until the required accuracy is achieved in the solution of the system for problems 1 and 2.

3. Calculation of Wave Fields in a Medium. To calculate the wave field in an elastic layer ($-b \le x \le 0$), we have representation (1.4). Inserting the solution of the integral equations (Eqs. (2.1) and (2.2) for problems 1 and 2, respectively) into (1.4), we reduce the problem to the calculation of a simple integral. This calculation can be done directly on a computer. To calculate the wave field in an elastic half-space with a cavity, for problem 1 we obtain

$$w_1^{(1)}(x, y) = -\frac{iap}{\pi\mu_1\theta_1 H_{-1}^{(1)}(\theta_1)}\int_\Gamma \frac{\exp\left(-\sigma_1(x+\varepsilon^{-1}) - i\alpha y\right)D^-}{\sigma_1 D^+}\, d\alpha + O\left(\frac{\varepsilon}{\theta_1}\right),$$
$$w_2^{(1)}(r, \varphi) = \frac{p}{\theta_1\mu_1}\left[\frac{H_0^{(1)}(\theta_1 r)}{H_{-1}^{(1)}(\theta_1)} + \sum_{m=-\infty}^{\infty}\frac{H_m^{(1)}(\theta_1 r)}{H_{-1}^{(1)}(\theta_1)\,\Delta_m(\theta_1)}\times\right.$$
$$\left.\times\, e^{i(m\varphi+2\theta_1\varepsilon^{-1}-\pi/4+m\pi/2)}\sqrt{\frac{\varepsilon}{\pi\theta_1}}\, J_m'(\theta_1)\, D_0^+/D_0^-\right] + O\left(\frac{\varepsilon}{\theta_1}\right).$$

Similarly, for problem 2

$$w_1^{(1)}(x, y) = -\frac{a}{2\pi\mu_1}\left\{\int\limits_\Gamma \frac{\exp(-\sigma_1 x - i\alpha y)}{\sigma_1}\left(\frac{\mu_1\sigma_1\tilde{Z}(\alpha)}{D^+} + 4\sqrt{\frac{\varepsilon}{2\pi\theta_1}}\times\right.\right.$$

$$\times \exp\left(i\left(\varepsilon^{-1}\theta_1 + \pi/4\right) - \varepsilon^{-1}\sigma_1\right) p(d-c)\left\{J_0'(\theta_1)/\left(2H_0'^{(1)}(\theta_1)\right) + \sum_{m=1}^{\infty} J_m'(\theta_1)\times\right.$$

$$\left.\left.\left.\times T_m\left(\frac{\sqrt{\theta_1^2 - \alpha^2}}{\theta_1}\right)\Big/ H_m'^{(1)}(\theta_1)\right\} D^-/(D_0^- D^+)\right) d\alpha\right\} + O\left(\frac{\varepsilon}{\theta_1}\right),$$

$$\overline{w}_2^{(1)}(r, \varphi) = -\sqrt{\frac{2\pi\varepsilon}{\theta_1}}\exp\left(i\theta_1\varepsilon^{-1} + i\pi/4\right) p(d-c)\left\{\frac{H_0^{(1)}(\theta_1 r)}{H_0'^{(1)}(\theta_1)} J_0'(\theta_1)\times\right.$$

$$\left.\times 2\sum_{m=1}^{\infty}\frac{H_m^{(1)}(\theta_1 R)}{H_m'^{(1)}(\theta_1)} J_m'(\theta_1)\, e^{im\pi/2}\cos m\varphi\right\}\Big/ D_0^-.$$

Figure 1 shows the behavior of the amplitude–frequency characteristic of the point of the medium with the coordinates x = y = 5 (ε = 0, 1, b = 1) for problem 2 with the following ratios: $\mu_1/\rho_1 = 1.32\cdot 10^5$, $\mu/\rho = 2.5\cdot 10^4$ — dashed line, $\mu_1/\rho_1 = 2.5\cdot 10^4$, $\mu/\rho = 1.32\cdot 10^5$ — the dot-dash line. The solid and dotted lines show characteristics of the same point in the case of a region without a cavity and analogous parameters of the problem. The qualitative difference between the graphs is due to the fact that for $V_S < V_{S1}$ ($V_S = \sqrt{\rho/\mu}$, $V_{S1} = \sqrt{\rho_1/\mu_1}$), the excitation of steady harmonic oscillations causes the layer to act as a waveguide along the boundaries of which propagate waves with a decreasing amplitude. These waves are due to the presence of real zeros with the function D^+. This condition is not determining for problem 1, since the oscillating cavity is mainly responsible for the formation of the wave field. The amplitude–frequency relation of a point of the medium during loading on the surface of a cylindrical hole is shown in Fig. 2. The correspondence between the lines and the parameters of the problem is the same as in Fig. 1.

Analysis of the behavior of the solution of problem 2 with respect to the angular coordinate with a fixed distance from the vibration source yielded the results shown in Fig. 3 in the form of the dependence of $w_1\mu_1$ on the angle ψ with $\theta_1 = 1$, $R_1 = \sqrt{x^2 + y^2} = 13$, $\psi = \operatorname{arctg} x/y$, $\varepsilon = 0.1$, $\rho_1 = \rho = 2\cdot 10^3$. The dashed line shows the dependence with $\mu_1 = 3\cdot 10^8$, $\mu = 7.5\cdot 10^7$, while the dot-dash line shows the dependence with $\mu_1 = 7.5\cdot 10^7$, $\mu = 3\cdot 10^8$. The solid and dotted lines show the analogous characteristics for a two-layer half-space without a cavity. An additional oscillation caused by the presence of the reflecting boundaries of the cavity, half-space, and layer is seen in the region with a hole.

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